

Càlcul d'integrals indefinides

Si $f : [a, b] \rightarrow \mathbb{R}$

S'anomena primitiva de $f(x)$ a tota funció $F(x)$ derivable en $[a, b]$ tal que $F'(x) = f(x)$.

S'anomena integral indefinida de la funció al conjunt $F(x) + C$ on $F(x)$ és una primitiva de $f(x)$. Ho denotarem per:

$$\int f(x) dx.$$

Propietats de les integrals indefinides:

- 1.- Si $\alpha \in \mathbb{R}$, $\int \alpha f(x) dx = \alpha \int f(x) dx$
- 2.- $\int (f + g)(x) dx = \int f(x) dx + \int g(x) dx$.

Integrals immediates

Tipus	Formes	
	Simplex	Compostes
Potencial $n \neq -1$	$\int k dx = kx + C$ $\int x^n dx = \frac{x^{n+1}}{n+1} + C$	$\int f^n \cdot f' dx = \frac{f^{n+1}}{n+1} + C$
Logarítmic	$\int \frac{1}{x} dx = \ln x + C$	$\int \frac{f'}{f} dx = \ln f + C$
Exponencial $a > 0, a \neq 1$	$\int e^x dx = e^x + C$ $\int a^x dx = \frac{a^x}{\ln a} + C$	$\int e^f \cdot f' dx = e^f + C$ $\int a^f \cdot f' dx = \frac{a^f}{\ln a} + C$
Cosinus	$\int \sin x dx = -\cos x + C$	$\int (\sin f) f' dx = -\cos f + C$
Sinus	$\int \cos x dx = \sin x + C$	$\int (\cos f) f' dx = \sin f + C$
Tangent	$\int (1 + \operatorname{tg}^2 x) dx = \operatorname{tg} x + C$ $\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$	$\int (1 + \operatorname{tg}^2 f) f' dx = \operatorname{tg} f + C$ $\int \frac{f'}{\cos^2 f} dx = \operatorname{tg} f + C$
Cotangent	$\int (1 + \operatorname{ctg}^2 x) dx = -\operatorname{ctg} x + C$ $\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$	$\int (1 + \operatorname{ctg}^2 f) f' dx = -\operatorname{ctg} f + C$ $\int \frac{f'}{\sin^2 f} dx = -\operatorname{ctg} f + C$
Arc sinus	$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$	$\int \frac{f'}{\sqrt{1-f^2}} dx = \arcsin f + C$
Arc tangent	$\int \frac{1}{1+x^2} dx = \operatorname{arctg} x + C$	$\int \frac{f'}{1+f^2} dx = \operatorname{arctg} f + C$

Tipus potencial

$$\int x^4 dx$$

$$\int 3x^5 dx$$

$$\int (2x^3 - 4x^2 + 1) dx$$

$$\int \frac{1}{x^3} dx$$

$$\int \sqrt[3]{x^2} dx$$

$$\int (x+1)^2 dx$$

$$\int (2x+3)^5 dx$$

$$\int (x^2 + x + 1)^4 (2x + 1) dx$$

$$\int \sin^3 x \cdot \cos x dx$$

$$\int \frac{\ln x}{x} dx$$

Tipus logarítmic

$$\int \frac{3}{x} dx$$

$$\int \frac{-4}{5x} dx$$

$$\int \frac{3x^2 + 1}{x^3 + x + 5} dx$$

$$\int \frac{x}{1+x^2} dx$$

$$\int \frac{1}{ax+b} dx$$

$$\int \frac{1}{x \cdot \ln x} dx$$

$$\int \operatorname{tg} x dx$$

Tipus exponencial

$$\int 3^x dx$$

$$\int \frac{3^x}{2^x} dx$$

$$\int 5xe^{x^2} dx$$

$$\int e^{\sin x} \cos x dx$$

$$\int 8^{2x+1} dx$$

$$\int \frac{4^{\arctg x}}{1+x^2} dx$$

Tipus sinus-cosinus

$$\int 3 \cos x dx$$

$$\int -5 \sin x dx$$

$$\int \frac{\sin 3x}{8} dx$$

$$\int \cos(5x+1) dx$$

$$\int e^x \sin(e^x) dx$$

$$\int x^3 \cos(2x^4 + 1) dx$$

Tipus tangent-cotangent

$$\int \frac{8}{\sin^2 x} dx$$

$$\int (5 + 5 \operatorname{tg}^2 x) dx$$

$$\int \frac{x}{\cos^2 x^2} dx$$

$$\int \operatorname{tg}^2 x dx$$

$$\int \operatorname{tg}^2 3x dx$$

Tipus arc sinus

$$\int \frac{4}{\sqrt{1-x^2}} dx$$

$$\int \frac{x}{\sqrt{1-x^4}} dx$$

$$\int \frac{e^x}{\sqrt{1-e^{2x}}} dx$$

$$\int \frac{1}{x\sqrt{1-\ln^2 x}} dx$$

$$\int \frac{1}{\sqrt{x}\sqrt{1-x}} dx$$

$$\int \frac{x^2}{\sqrt{1-x^6}} dx$$

Tipus arc tangent

$$\int \frac{1}{3+3x^2} dx$$

$$\int \frac{1}{1+9x^2} dx$$

$$\int \frac{x}{1+x^4} dx$$

$$\int \frac{\cos x}{1+\sin^2 x} dx$$

$$\int \frac{e^x}{1+e^{2x}} dx$$

$$\int \frac{1}{a^2+x^2} dx$$

$$\int \frac{x^2}{1+4x^6} dx$$

Integrals per substitució:

Moltes vegades fent el canvi $x = \theta(t)$ la integral es transforma en una d'altra més senzilla.

Aleshores,
 $dx = \theta'(t) dt$

$$\int f(x) dx = \int f(\theta(t)) \cdot \theta'(t) dt$$

Al final es desfà el canvi substituint t per $\theta^{-1}(x)$.

Problema 1:

$$\text{Calculeu } \int x^2 \sqrt{x-7} dx$$

Solució:

Efectuem el canvi $t = \sqrt{x-7}$, $x = 7 + t^2$
 $dx = 2t dt$.

$$\begin{aligned} \int x^2 \sqrt{x-7} dx &= \int (7 + t^2)^2 \cdot t \cdot 2t dt = \int (2t^6 + 28t^4 + 98t^2) dt = \\ &= \frac{2}{7} t^7 + \frac{28}{5} t^5 + \frac{98}{3} t^3 + C = \\ &= \frac{2}{7} \sqrt{(x-7)^7} + \frac{28}{5} \sqrt{(x-7)^5} + \frac{98}{3} \sqrt{(x-7)^3} + C. \end{aligned}$$

Problema 2:

$$\text{Calculeu } \int \sqrt{16-x^2} dx$$

Solució:

Efectuem el canvi, $x = 4 \sin t$ a fi que desaparega l'arrel.

$$dx = 4 \cos t dt, \quad t = \arcsin\left(\frac{x}{4}\right).$$

$$\begin{aligned} \int \sqrt{16-x^2} dx &= \int \sqrt{16-(4 \sin t)^2} 4 \cos t \cdot dt = \\ &= \int \sqrt{16(1-\sin^2 t)} 4 \cos t dt = \\ &= \int 16 \cos^2 t dt = \end{aligned}$$

$$\text{Recordem que } \cos^2 t = \frac{1}{2}(1 + \cos 2t)$$

$$\begin{aligned} &= \int 8 \frac{1}{2} (1 + \cos 2t) dt = \\ &= 8 \left(\int dt + \frac{1}{2} \int 2 \cos 2t dt \right) = \\ &= 8t + 4 \sin 2t + C = \\ &= 8 \arcsin\left(\frac{x}{4}\right) + 4 \sin\left(2 \arcsin\left(\frac{x}{4}\right)\right) + C. \end{aligned}$$

Exercicis Integrals per substitució:

$$\int \frac{1}{(1+x)\sqrt{x}} dx$$

$$\int \frac{x}{\sqrt{1+x}} dx$$

$$\int x\sqrt{1+x} dx$$

$$\int \sqrt{e^x - 1} dx$$

$$\int x\sqrt{3+2x} dx$$

$$\int \frac{e^x}{e^x + 1} dx$$

$$\int \frac{e^{2x}}{e^x + 1} dx$$

$$\int \frac{e^x}{e^{2x} + 1} dx$$

Integració per parts

$$d(uv) = d(u) \cdot v + u \cdot d(v)$$

Aleshores,

$$uv = \int d(uv) = \int v \cdot d(u) + \int u \cdot d(v)$$

Per tant,

$$\int u \cdot d(v) = uv - \int v \cdot d(u)$$

Aleshores el mètode s'aplicarà sempre que $\int v \cdot d(u)$ tinga menys dificultat de càlcul integral que la integral de partida i que $d(v)$ siga fàcilment integrable.

Problema:

$$\text{Calculeu } \int xe^x dx.$$

Solució:

$$\text{Fent el canvi } u = x, \quad d(u) = 1 \cdot dx$$

$$d(v) = e^x dx, \quad v = \int e^x dx = e^x.$$

Aplicant el mètode:

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C = (x-1)e^x + C.$$

Problema:

$$\text{Calculeu } I = \int e^x \sin x dx.$$

Solució:

$$\text{Fent el canvi } u = e^x \quad d(u) = e^x dx$$

$$d(v) = \sin x dx, \quad v = \int \sin x dx = -\cos x.$$

Aplicant el mètode:

$$I = \int e^x \sin x dx = -\cos x \cdot e^x - \int (-\cos x)e^x dx = -\cos x \cdot e^x + \int \cos x \cdot e^x dx$$

$$\text{Fem el canvi } u = e^x \quad d(u) = e^x dx$$

$$d(v) = \cos x dx, \quad v = \int \cos x dx = \sin x.$$

Per tant, aplicant el mètode:

$$\int \cos x \cdot e^x dx = e^x \cdot \sin x - \int e^x \sin x dx.$$

Aleshores,

$$I = \int e^x \sin x dx = -\cos x \cdot e^x + \int \cos x \cdot e^x dx = -\cos x \cdot e^x + \sin x \cdot e^x - I$$

Resolem l'equació en la incògnita I:

$$2I = -\cos x \cdot e^x + \sin x \cdot e^x$$

Aleshores,

$$I = \frac{e^x(\sin x - \cos x)}{2} + C.$$

Exercicis integració per parts:

$$\int \ln x \, dx$$

$$\int x \cdot \cos x \, dx$$

$$\int x^2 e^x \, dx$$

$$\int x \cdot \ln x \, dx$$

$$\int e^x \cos x \, dx$$

$$\int \arctg x \, dx$$

$$\int e^{2x} \sin x \, dx$$

$$\int e^x \cos 3x \, dx$$

$$\int \sin^4 x \, dx$$

$$\int x \cdot 7^x \, dx$$

$$\int \frac{\ln x}{x^3} dx$$

Integració de funcions racionals

Siga $\int \frac{p(x)}{q(x)} dx$, tal que $p(x), q(x)$ són polinomis.

Es pot suposar que el grau de $p(x)$ es menor que el grau de $q(x)$.

Si el grau de $p(x)$ és major o igual que el grau de $q(x)$, realitzariem prèviament la divisió:

$\frac{p(x)}{q(x)} = h(x) + \frac{r(x)}{q(x)}$, tal que $h(x)$, $r(x)$ són polinomis i grau de $r(x)$ es menor que el grau de $q(x)$.

Siga $p(x)$, $q(x)$ dos polinomis tal que el grau de $p(x)$ és major o igual que el grau de $q(x)$.

Si coneixem les arrels del denominador $q(x)$ és poden plantejar 4 situacions:

- Arrels reals simples.
- Arrels reals múltiples.
- Arrels complexes (Aquest cas no l'estudiarem).

a) Arrels reals simples

Si les arrels de $q(x)$ són x_1, x_2, \dots, x_n reals i distintes.

$$q(x) = (x - x_1)(x - x_2) \cdot \dots \cdot (x - x_n).$$

Aleshores,:

$$\frac{p(x)}{q(x)} = \frac{A_1}{x - x_1} + \frac{A_2}{x - x_2} + \dots + \frac{A_n}{x - x_n} \text{ on } A_1, A_2, \dots, A_n \in \mathbb{R}.$$

Per tant:

$$\begin{aligned} \int \frac{p(x)}{q(x)} dx &= \int \frac{A_1}{x - x_1} dx + \int \frac{A_2}{x - x_2} dx + \dots + \int \frac{A_n}{x - x_n} dx = \\ &= A_1 \ln|x - x_1| + A_2 \ln|x - x_2| + \dots + A_n \ln|x - x_n| + C \end{aligned}$$

Problema

Calculeu $\int \frac{1}{x^2 - 2x - 3} dx$

Solució:

El grau del numerador és menor que el grau del denominador.

Les arrels del denominador són $x = -1, 3$

$$x^2 - 2x - 3 = (x + 1)(x - 3).$$

$$\frac{1}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}. \text{ Hem de determinar } A, B \in \mathbb{R}$$

$$1 = A(x - 3) + B(x + 1).$$

$$1 = (A + B)x - 3A + B.$$

Identificant coeficients dels polinomis ens queda el següent sistema:

$$\begin{cases} A + B = 0 \\ -3A + B = 1 \end{cases}, \text{ la solució del qual és, } \begin{cases} A = -\frac{1}{4} \\ B = \frac{1}{4} \end{cases} \text{ Aleshores:}$$

$$\int \frac{1}{x^2 - 2x - 3} dx = \frac{-1}{4} \int \frac{1}{x+1} dx + \frac{1}{4} \int \frac{1}{x-3} dx = \frac{-1}{4} \ln|x+1| + \frac{1}{4} \ln|x-3| + C.$$

Problema:

Calculeu $\int \frac{x^3 + 9x^2 + 23x + 17}{x^3 + 6x^2 + 11x + 6} dx$

Solució:

El grau del numerador no és menor que el grau del denominador. Efectuem la divisió:

$$\frac{x^3 + 9x^2 + 23x + 17}{x^3 + 6x^2 + 11x + 6} = 1 + \frac{3x^2 + 12x + 11}{x^3 + 6x^2 + 11x + 6}$$

Calculem les arrels del denominador amb la regla de Ruffini:

$x = -1, -2, -3$, aleshores:

$$x^3 + 6x^2 + 11x + 6 = (x+1)(x+2)(x+3)$$

$$\frac{3x^2 + 12x + 11}{x^3 + 6x^2 + 11x + 6} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}. \text{ Hem de determinar } A, B, C \in \mathbb{R}.$$

$$3x^2 + 12x + 11 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

Per a $x = -1$,

$$3(-1)^2 + 12(-1) + 11 = A(-1+2)(-1+3) + B(-1+1)(-1+3) + C(-1+1)(-1+2)$$

$$2 = 2A, \text{ Per tant, } A = 1.$$

Per a $x = -2$, $-1 = -B$, Per tant, $B = 1$.

Per a $x = -3$, $2 = 2C$, Per tant, $C = 1$.

Aleshores,

$$\begin{aligned} \int \frac{x^3 + 9x^2 + 23x + 17}{x^3 + 6x^2 + 11x + 6} dx &= \int \left(1 + \frac{3x^2 + 12x + 11}{x^3 + 6x^2 + 11x + 6} \right) dx = \\ &= \int \left(x + \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} \right) dx = \\ &= \frac{x^2}{2} + \ln|x+1| + \ln|x+2| + \ln|x+3| + C \end{aligned}$$

Exercicis d'integrals racionals arrels reals simples

$$\int \frac{x+1}{x^2 - 2x} dx$$

$$\int \frac{x^3 + 1}{x^2 - 4} dx$$

$$\int \frac{1}{x(x+2)(x-1)} dx$$

$$\int \frac{x^2 + 1}{x^3 - 7x - 6} dx$$

b) Arrels reals múltiples

Si les arrels de $q(x)$ són x_1, x_2, \dots, x_n reals de multiplicitat r_1, r_2, \dots, r_n , respectivament.

$$q(x) = (x - x_1)^{r_1} (x - x_2)^{r_2} \cdot \dots \cdot (x - x_n)^{r_n}.$$

Aleshores,:

$$\frac{p(x)}{q(x)} = \frac{A_{11}}{x - x_1} + \frac{A_{12}}{(x - x_1)^2} + \dots + \frac{A_{1r_1}}{(x - x_1)^{r_1}} + \dots + \frac{A_{n1}}{x - x_n} + \frac{A_{n2}}{(x - x_n)^2} + \dots + \frac{A_{nr_n}}{(x - x_n)^{r_n}} \quad \text{on}$$

$$A_{ij} \in \mathbb{R}.$$

Per tant:

$$\begin{aligned} \int \frac{p(x)}{q(x)} dx &= \int \frac{A_{11}}{x - x_1} dx + \int \frac{A_{12}}{(x - x_1)^2} dx + \dots + \int \frac{A_{1r_1}}{(x - x_1)^{r_1}} dx + \dots + \\ &+ \dots + \int \frac{A_{n1}}{x - x_n} dx + \int \frac{A_{n2}}{(x - x_n)^2} dx + \dots + \int \frac{A_{nr_n}}{(x - x_n)^{r_n}} dx = \\ &= A_{11} \ln|x - x_1| + A_{12} \frac{-1}{x - x_1} + \dots + A_{1r_1} \frac{1}{(1 - r_1)(x - x_1)^{r_1 - 1}} + \dots + \\ &+ \dots + A_{n1} \ln|x - x_n| + A_{n2} \frac{-1}{x - x_n} + \dots + A_{nr_n} \frac{1}{(1 - r_n)(x - x_n)^{r_n - 1}} + C \end{aligned}$$

Problema:

$$\text{Calculeu } \int \frac{3x + 1}{(x + 1)^2} dx$$

Solució:

El grau del numerador és menor que el grau del denominador.

L'arrel del denominador són $x = -1$ de multiplicitat 2.

$$\frac{3x + 1}{(x + 1)^2} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2}. \quad \text{Hem de determinar } A, B \in \mathbb{R}.$$

$$3x + 1 = A(x + 1) + B$$

$3x + 1 = Ax + A + B$. Igualant els coeficients dels polinomis:

$$\begin{cases} A = 3 \\ A + B = 1 \end{cases}, \quad \text{la solució del sistema és: } \begin{cases} A = 3 \\ B = -2 \end{cases}.$$

Aleshores,

$$\int \frac{3x + 1}{(x + 1)^2} dx = \int \frac{3}{x + 1} dx + \int \frac{-2}{(x + 1)^2} dx = 2 \ln|x + 1| + \frac{2}{x - 1} + C.$$

Exercicis d'integrals racionals arrels reals múltiples

$$\int \frac{x}{x^2 - 4x + 4} dx$$

$$\int \frac{1}{x^4 + 4x^3 + 4x^2} dx$$

$$\int \frac{x + 1}{(x^2 - 4)^2} dx$$

$$\int \frac{2x+1}{x^3-6x^2+9x} dx$$

$$\int \frac{1}{(x+1)^2(x-3)^2} dx$$

Racionals amb canvi de variable

$$\int \frac{e^{2x}}{1+e^x} dx$$

$$\int \frac{x}{1-\sqrt{x}} dx$$

$$\int \frac{2^{2x}+1}{2^x-1} dx$$

$$\int \frac{x^2}{\sqrt{1+x}} dx$$

$$\int \frac{x}{1+\sqrt{x}} dx$$

$$\int \frac{x^3}{\sqrt{x-1}} dx$$

$$\int \frac{\sqrt{x-1}}{1-\sqrt{x-1}} dx$$

Soluciones

Tipus potencial

$$\int x^4 dx = \frac{x^5}{5} + C \quad \int 3x^5 dx = \frac{1}{2}x^6 + C \quad \int (2x^3 - 4x^2 + 1)dx \quad \int \frac{1}{x^3} dx = \frac{-1}{2x^2} + C$$
$$\int \sqrt[3]{x^2} dx = \frac{3}{5}\sqrt[3]{x^5} + C \quad \int (x+1)^2 dx = \frac{(x+1)^3}{3} + C \quad \int (2x+3)^5 dx = \frac{1}{12}(2x+3)^6 + C$$
$$\int (x^2 + x + 1)^4 (2x+1) dx = \frac{(x^2 + x + 1)^5}{5} + C \quad \int \sin^3 x \cdot \cos x dx = \frac{\sin^4 x}{4} + C$$
$$\int \frac{\ln x}{x} dx = \frac{\ln^2 x}{2} + C$$

Tipus logarímic

$$\int \frac{3}{x} dx = 3\ln|x| + C \quad \int \frac{-4}{5x} dx = \frac{-4}{5}\ln|x| + C \quad \int \frac{3x^2 + 1}{x^3 + x + 5} dx = \ln|x^3 + x + 5| + C$$
$$\int \frac{x}{1+x^2} dx = \frac{1}{2}\ln|1+x^2| + C \quad \int \frac{1}{ax+b} dx = \frac{1}{a}\ln|ax+b| + C \quad \int \frac{1}{x \cdot \ln x} dx = \ln|\ln|x|| + C$$
$$\int \operatorname{tg} x dx = -\ln|\cos x| + C$$

Tipus exponencial

$$\int 3^x dx = \frac{3^x}{\ln 3} + C \quad \int \frac{3^x}{2^x} dx = \frac{\left(\frac{3}{2}\right)^x}{\ln \frac{3}{2}} + C \quad \int 5xe^{x^2} dx = \frac{5}{2}e^{x^2} + C$$
$$\int e^{\sin x} \cos x dx = e^{\sin x} + C \quad \int 8^{2x+1} dx = \frac{1}{2} \frac{8^{2x+1}}{\ln 8} + C \quad \int \frac{4^{\operatorname{arctg} x}}{1+x^2} dx = \frac{4^{\operatorname{arctg} x}}{\ln 4} + C$$

Tipus sinus-cosinus

$$\int 3\cos x dx = 3\sin x + C \quad \int -5\sin x dx = 5\cos x + C \quad \int \frac{\sin 3x}{8} dx = \frac{-1}{24}\cos 3x + C$$
$$\int \cos(5x+1) dx = \frac{1}{5}\cos(5x+1) + C \quad \int e^x \sin(e^x) dx = -\cos(e^x) + C$$
$$\int x^3 \cos(2x^4 + 1) dx = \frac{1}{8}\sin(2x^4 + 1) + C$$

Tipus tangent-cotangent

$$\int \frac{8}{\sin^2 x} dx = 8 - \operatorname{ctg} x + C \quad \int (5 + 5\operatorname{tg}^2 x) dx = 5\operatorname{tg} x + C \quad \int \frac{x}{\cos^2 x^2} dx = \frac{1}{2}\operatorname{tg} x^2 + C$$
$$\int \operatorname{tg}^2 x dx = -x + \operatorname{tg} x + C \quad \int \operatorname{tg}^2 3x dx = -x + \frac{1}{3}\operatorname{tg}(3x) + C$$

Tipus arc sinus

$$\int \frac{4}{\sqrt{1-x^2}} dx = 4\arcsin x + C \quad \int \frac{x}{\sqrt{1-x^4}} dx = \frac{1}{2}\arcsin x^2 + C$$
$$\int \frac{e^x}{\sqrt{1-e^{2x}}} dx = \arcsin(e^x) + C \quad \int \frac{1}{x\sqrt{1-\ln^2 x}} dx = \arcsin(\ln x) + C$$
$$\int \frac{1}{\sqrt{x}\sqrt{1-x}} dx = 2\arcsin\sqrt{x} + C \quad \int \frac{x^2}{\sqrt{1-x^6}} dx = \frac{1}{3}\arcsin x^3$$

Tipus arc tangent

$$\int \frac{1}{3+3x^2} dx = \frac{1}{3} \operatorname{arctg} x + C \quad \int \frac{1}{1+9x^2} dx = \frac{1}{3} \operatorname{arctg}(3x) + C$$
$$\int \frac{x}{1+x^4} dx = \frac{1}{2} \operatorname{arctg} x^2 + C \quad \int \frac{\cos x}{1+\sin^2 x} dx = \operatorname{arctg}(\sin x) + C$$
$$\int \frac{e^x}{1+e^{2x}} dx = \operatorname{arctg}(e^x) + C \quad \int \frac{1}{a^2+x^2} dx = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C$$
$$\int \frac{x^2}{1+4x^6} dx = \frac{1}{6} \operatorname{arctg}(2x^3) + C$$

Exercicis Integrals per substitució

$$\int \frac{1}{(1+x)\sqrt{x}} dx = 2 \operatorname{arctg} \sqrt{x} + C \quad \int \frac{x}{\sqrt{1+x}} dx = \frac{2}{3} (x+2)\sqrt{1+x} + C$$
$$\int x\sqrt{1+x} dx = \frac{2}{15} (3x-2)\sqrt{(1+x)^3} + C \quad \int \sqrt{e^x-1} dx = 2\sqrt{e^x-1} - 2 \operatorname{arctg} \sqrt{e^x-1} + C$$
$$\int x\sqrt{3+2x} dx = \frac{-1}{5} (x+1)\sqrt{(3-2x)^3} + C \quad \int \frac{e^x}{e^x+1} dx = \ln|e^x+1| + C$$
$$\int \frac{e^{2x}}{e^x+1} dx = e^x - \ln|e^x+1| + C \quad \int \frac{e^x}{e^{2x}+1} dx = \operatorname{arctg}(e^x) + C$$

Exercicis integració per parts:

$$\int \ln x dx = x(-1+\ln|x|) + C \quad \int x \cdot \cos x dx = \cos x + x \cdot \sin x + C$$
$$\int x^2 e^x dx = e^x(x^2 - 2x + 2) + C \quad \int x \cdot \ln x dx = \frac{x^2}{4}(-x^2 + \ln|x|) + C$$
$$\int e^x \cos x dx = \frac{e^x}{2}(\cos x + \sin x) + C \quad \int \operatorname{arctg} x dx = x \cdot \operatorname{arctg} x - \frac{1}{2} \ln|1+x^2| + C$$
$$\int e^{2x} \sin x dx = \frac{e^{2x}}{5}(2 \sin x - \cos x) + C \quad \int e^x \cos 3x dx = \frac{e^x}{10}(\cos(3x) + 3 \sin(3x)) + C$$
$$\int \sin^4 x dx = \frac{3}{8}x - \frac{1}{8} \cos x(2 \sin^3 x + 3 \sin x) + C \quad \int x \cdot 7^x dx = \frac{7^x}{\ln^2 7}(-1 + x \cdot \ln 7) + C$$
$$\int \frac{\ln x}{x^3} dx = \frac{-1}{4x^2}(1 + 2 \ln|x|) + C$$

Exercicis d'integrals racionals arrels reals simples

$$\int \frac{x+1}{x^2-2x} dx = \frac{3}{2} \ln|x-2| - \frac{1}{2} \ln|x| + C \quad \int \frac{x^3+1}{x^2-4} dx = \frac{1}{2}x^2 + \frac{7}{4} \ln|x+2| + \frac{9}{4} \ln|x-2| + C$$
$$\int \frac{1}{x(x+2)(x-1)} dx = \frac{-1}{2} \ln|x| + \frac{1}{6} \ln|x+2| + \frac{1}{3} \ln|x-1| + C$$
$$\int \frac{x^2+1}{x^3-7x-6} dx = \frac{1}{2} \ln|x-3| + \ln|x+2| + \frac{-1}{2} \ln|x+1| + C$$

Exercicis d'integrals racionals arrels reals múltiples

$$\int \frac{x}{x^2-4x+4} dx = \ln|x-2| + \frac{2}{x-2} + C$$
$$\int \frac{1}{x^4+4x^3+4x^2} dx = \frac{-1}{4} \ln|x| + \frac{-1}{4x} + \frac{1}{4} \ln|x+2| + \frac{-1}{4(x+2)} + C$$
$$\int \frac{x+1}{(x^2-4)^2} dx = \frac{1}{32} \ln|x+2| + \frac{1}{16(x+2)} - \frac{1}{32} \ln|x-2| + \frac{-3}{16(x-2)} + C$$

$$\int \frac{2x+1}{x^3-6x^2+9x} dx = \frac{1}{9} \ln|x| + \frac{-1}{9} \ln|x-3| - \frac{7}{3(x-3)} + C$$

$$\int \frac{1}{(x+1)^2(x-3)^2} dx = \frac{1}{32} \ln|x+1| - \frac{1}{16(x+1)} - \frac{1}{32} \ln|x-3| - \frac{1}{16(x-3)} + C$$

Racionals amb canvi de variable

$$\int \frac{e^{2x}}{1+e^x} dx = e^x - \ln|1+e^x| + C \quad \int \frac{x}{1-\sqrt{x}} dx = -\frac{\sqrt{x}(2x+\sqrt{x}+6)}{3} - 2\ln|1-\sqrt{x}| + C$$

$$\int \frac{2^{2x}+1}{2^x-1} dx = -x + \frac{2^x}{\ln 2} + \frac{2\ln|2^x-1|}{\ln 2} + C \quad \int \frac{x^2}{\sqrt{1+x}} dx = \frac{2}{15} (3x^2-4x+8)\sqrt{x+1} + C$$

$$\int \frac{x}{1+\sqrt{x}} dx = -x + 2\sqrt{x} + \frac{2}{3}\sqrt{x^3} - 2\ln|1+\sqrt{x}| + C$$

$$\int \frac{x^3}{\sqrt{x-1}} dx = \frac{2}{35} (5x^3+6x^2+8x+16)\sqrt{x-1} + C$$

$$\int \frac{\sqrt{x-1}}{1-\sqrt{x-1}} dx = -x - 2\sqrt{x-1} - 2\ln|\sqrt{x-1}-1| + C$$